RRST-Mathematics

A Note on Kaehlerian Manifolds with Vanishing Bochner Curvature Tensor

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Article Info	Abstract
	The purpose of the paper is to prove a theorem in a Kaehlerian manifold M with vanishing Bochner curvature tensor replacing the condition of constancy of scalar curvature and the length of the Ricci Tensor by the harmonic length of the Ricci tensor.

Introduction

In this paper we consider a Kaehlerian Manifold M of real dimension n with almost complex structure tensor F of type (1, 1) and a Kaehlerian metric tensor G. Let us denote by g_{ji}, ∇_i, $K_{kji}{}^h$, K_{ji}, K and F_j^i local components of g , the operator of covariant differentiation with respect to Levi-Civita connection, the curvature tensor , the Ricci Tensor, the scalar curvature and F of M respectively. The indices h, i, j, k…run over the range 1,2…n.

The Bochner Curvature of M is defined [1] by

(1.1)

$$B_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki} + F_k{}^h M_{ji}$$
$$- F_j{}^h M_{ki} + M_k^h F_{ji} - M_j^h F_{ki} - 2(M_{kj} F_i^h + F_{kj} M_i^h),$$

where

$$L_{ji} = -\frac{1}{n+4} K_{ji} + \frac{1}{2(n+2)(n+4)} K g_{ji},$$

$$L_k^h = L_{kt} g^{th},$$

$$M_{ji} = -L_{jt} F_i^t = -\frac{1}{n+4} H_{ji} + \frac{1}{2(n+2)(n+4)} K F_{ji},$$

$$M_k^h = M_{kt} g^{th},$$

$$H_{ji} = -K_{jt} F_i^t.$$

Kaehlerian manifolds with vanishing Bochner Curvature tensor had been studied by many authors such as K. C. Nam[1], S . Tachibana [2], K. Yano and I. Shihara [3] and Y.

Kubo [4] and established the interesting properties of Kaehlerian manifolds. In 1976 Y. Kubo [4] has proved the following theorems:

Theorem A: Let M be Kaehlerian manifolds of real dimension n (≥ 4) with constant scalar curvature whose Bochner Curvature vanishes. If length of the Ricci Tensor is not greater than $\frac{K}{\sqrt{n-2}}$, then M is a space of constant holomorphic sectional curvature.

Theorem B: Let M be a Kaehlerian Manifolds of real dimension n (≥ 4) whose Bochner curvature vanishes. If the length of the Ricci tensor is constant and not greater than $\frac{K}{\sqrt{n-2}}$, then M is a space of constant holomorphic sectional curvature.

Following theorem due to K.C.Nam [1] is needed to the main results of the paper.

Theorem C (Nam[1]) **:** Let M be a Kaehlerian Manifolds with vanishing Bochner curvature Tensor whose length of Ricci tensor is constant. Then M is a space of constant holomorphic sectional curvature or a locally product of space of two spaces of constant holomorphic sectional curvature H (≥ 0).

The paper has been organized as follows. In section 1, known theorems have been given along with some basic notations and definitions of Bochner Curvature. In section 2, preliminary results of Kaehlerian manifolds and some known

lemmas have been given. Section 3 is devoted to study of the main theorems and some corollaries.

Preliminaries

In a Kaehlerian manifolds M, the local components F_j^i and g_{ji} of F and g are respectively satisfies,

(2.1)
$$F_j^t F_i^t g_{ts} = g_{ji},$$
$$\nabla_j F_i^h = 0,$$
and $F_j^t F_i^s K_{ts} = K_{ji}.$

If the Bochner Curvature Tensor vanishes, then it is proved that [3]

(2.2)
$$K_{kji}{}^h K_h^k K^{ji} = \frac{1}{n+4}[4K_t^s K_s^r K_r^t + \frac{2(n+1)}{n+2} KK_{ji}K^{ji} - \frac{1}{n+2}K^3]$$

Next we list the known Lemma for sake of easy reference in the next section 3.

Lemma A (K. Yano and Ishihara S [3]): In a Riemannian manifold of dimension n, for Q defined by

$$Q = nK_t^s K_s^r K_r^t - 2\frac{(n+1)}{(n+2)}[KK_{ji}K^{ji} + \frac{1}{n-2}K^3]$$

.

We have

(2.3)
$$Q = P + \frac{3n}{(n-1)(n+2)}K(K_{ji} - \frac{K}{n}g_{ji})(K^{ji} - \frac{K}{n}g^{ji})$$

where

$$P = nK_t^s K_s^r K_s^r - \frac{2n-1}{n-1}KK_{ji}K^{ji} + \frac{1}{n-1}K^3.$$

Lemma B(S. Tachibana [2]): If the Bochner Curvature tensor of a Kaehlerian manifold vanishes and the manifold is Eienstein manifold, then the Kaehlerian manifold is of constant holomorphic sectional curvature.

Lemma C(M .Okumura[5]): Let $a_i, i = 1,2,3.........n.$ be n real numbers satisfying

$$\sum_{i=1}^{n} a_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} a_i^2 = k^2,$$

for certain k. Then we have

$$-\frac{n-2}{\sqrt{n(n-1)}}k^3 \le \sum_{i=1}^{n} a_i^3 \le \frac{n-2}{\sqrt{n(n-1)}}k^3.$$

Lemma D(Y.Kubo[4]): Put

$$S_{ji} = K_{ji} - \frac{K}{n}g_{ji}, f^2 = S_{ji}S^{ji}$$

and a_i, i= 1,2,..., n be eigen values of S_j^i, then we have

$$-\frac{(n-4)}{\sqrt{2n(n-2)}}f^3 \le \sum_{i=1}^{n} a_i^3 \le \frac{(n-4)}{\sqrt{2n(n-2)}}f^3.$$

Towards this end we consider the concluding section of the paper.

Main Theorems

As stated in the introduction we prove the main theorems and corollaries.

Theorem3.1: Let M be a Kaehlerian manifold of real dimension $n(\ge 4)$ whose Bochner curvature tensor vanishes. If

i) $(\nabla_h K_{ji})(\nabla^h K^{ji}) + \frac{1}{2(n+2)}[(n+4)K_{ji}+Kg_{ji}]\nabla^j\nabla^i K=0$

and ii) $\Delta l = 0$ where $l \le \frac{K}{\sqrt{n-2}}$, $l^2 = K_{ji}K^{ji}$

then M is a space of constant holomorphic sectional curvature.

Proof : By the straight forward calculations ,we obtain

(3.1)
$$\frac{1}{2}\Delta(K_{ji}K^{ji}) = (\nabla_h K_{ji})(\nabla^h K_{ji})$$
$$+\frac{1}{2}K^{ji}\nabla_j\nabla_i K + K^{ji}K_{rj}K_i^r$$
$$-K_{kjih}K^{kh}K^{ji} + K^{ji}\nabla^r(\nabla_r K_{ji} - \nabla_i K_{rj})$$

For the sack of easy reference, we put

(3.2)
$$B_{kji} = \nabla_k K_{ji} - \nabla_j K_{ki}$$
$$+\frac{1}{2(n+2)}(g_{ki}\delta_j^a - g_{ji}\delta_k^a + F_{ki}F_j^a - F_{ji}F_k^a + 2F_{kj}F_i^a)\nabla_a K$$

next by straight forward calculation, we get [2]

(3.3)
$$\nabla_a B_{kji}{}^a = \frac{n}{(n+4)}B_{kji}$$

From (3.2) and (3.3) and the assumption that the Bochner curvature tensor vanishes, we have

(3.4)
$$\frac{1}{2}\Delta(K_{ji}K^{ji}) = Q + (\nabla_h K_{ji})(\nabla^h K^{ji})$$
$$+\frac{1}{2(n+2)}\{(n+4)K_{ji} + Kg_{ji}\}\nabla^j\nabla^i K$$

If we set on M,

$$S_{ji} = K_{ji} - \frac{K}{n}g_{ji},$$

then we have [4]

$$S_j^i F_i^l = F_j^i S_i^l,$$

Moreover , we see that [4]
(3.5)

$$traceS = S_i^i = 0$$

$$traceS^2 = S_{ji}S^{ji} = K_{ji}K^{ji} - \frac{K^2}{n} \geq 0$$

$$traceS^3 = S_j^i S_i^h S_h^j = K_j^i K_i^h K_h^j - \frac{3}{n}KS_{ji}S^{ji} - \frac{K^3}{n^2}$$

In (3.5), the equality holds if and only if M is an Einstein manifold.

Put $f^2 = trace \ S^2$ then, from Lemma C, we have

$$-\frac{(n-4)}{\sqrt{2n(n-2)}}f^3 \leq \sum_{i=1}^n a_i^3 \leq \frac{(n-4)}{\sqrt{2n(n-2)}}f^3$$

where, a_i , i=1,2,3... n be eigen values of S_j^i .

By the virtue of Lemma D , we have
(3.6)

$$P \geq \frac{n(n-4)}{\sqrt{2n(n-2)}}f^3 + \frac{(n-2)}{(n-1)}Kf^2 = f^2\left\{\frac{n-2}{n-1}K - \frac{n(n-4)}{\sqrt{2n(n-2)}}f\right\} \geq 0$$

So that $P \geq 0$ from (3.6) and the fact that

$$l \leq \frac{K}{\sqrt{n-2}}, \qquad n \geq 4$$

Since $\Delta l = 0$ i.e. l is harmonic together with the condition (i) of the theorem and (3.4),
we have
Q=0
Using (3.5) and (3.6) in Lemma A, we have

$$P \leq 0$$

And therefore P = 0, so that (2.3) becomes

(3.7) $\quad K(K_{ji} - \frac{K}{n}g_{ji})(K^{ji} - \frac{K}{n}g^{ji}) = 0$.

If $K \neq 0$, (3.7) gives

$$K_{ji} = \frac{K}{n}g_{ji} \qquad or \qquad S_{ji} = 0$$

That is, M is an Einstein Manifold, and hence from Lemma B, conclusion of the theorem follows. If K = 0, then

$K_{ji}K^{ji} = S_{ji}S^{ji} = 0$ .So that $\quad K_{ji} = 0 \quad$ and

consequently Kaehlerian manifold of zero curvature . Finally proof of the theorem completes.

Theorem 3.2: Let M be a compact orientable   Kaehlerian Manifold of real dimension $n(\geq 4)$ whose Bochner curvature tensor vanishes if the condition (i) of theorem A together with $\Delta l \geq 0$ (or $\Delta l \leq 0$)holds   where

$$l \leq \frac{K}{\sqrt{n-2}}, \qquad l^2 = K_{ji}K^{ji},$$

then M is a space of constant holomorphic sectional curvature .

Proof: If $\Delta l \geq 0$, then l is said to be superharmonic in M. For a compact orientable Kaehlerian manifold, we have Yano [6] $\Delta l = 0$ and l must be constant in M. Hence from the similar arguments as in Theorem 3.1 proof of Theorem 3.2 follows.

Corollary 3.1: Let M be a compact orientable Kaehlerian manifold of real dimension n $(n \geq 4)$ with constant scalar curvature whose Bochner curvature vanishes. If l is harmonic and $l \leq \frac{K}{\sqrt{n-2}}$, then M is a space holomorphic sectional curvature.

Corollary 3.2: Let M be a compact orientable Kaehlerian manifold of real dimension n $(n \geq 4)$ with constant scalar curvature whose Bochner curvature tensor vanishes. If l is constant and not greater than $\frac{K}{\sqrt{n-2}}$, then M is a space constant holomorphic sectional curvature.

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